ALM - Basic Interest Rate Theory

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Course Program

- `Basic interest rate theory`
- Interest rate risk management
- Stochastic term structure models
- Risk measurement
- Reinsurance and insurance-linked securities
- Mean-variance analysis for ALM

Contents of the chapter

- A continuous model for yield curves.
- Estimating the yield curve.
- Sensitivity of present values.

Definition of yield

- If $P(t)$ is the market price of a "zero-coupon bond" that pays the risk-free amount of €1 at time t, its yield y is defined by the equation:

$$P(t) = e^{-yt}$$

- The yield of the zero-coupon bond is defined as:

$$y(t) = -\frac{1}{t} ln\left(P(t)\right)$$

- $y(t)$ is called the "spot rate" or "zero rate" for maturity t.

The chicken and the egg

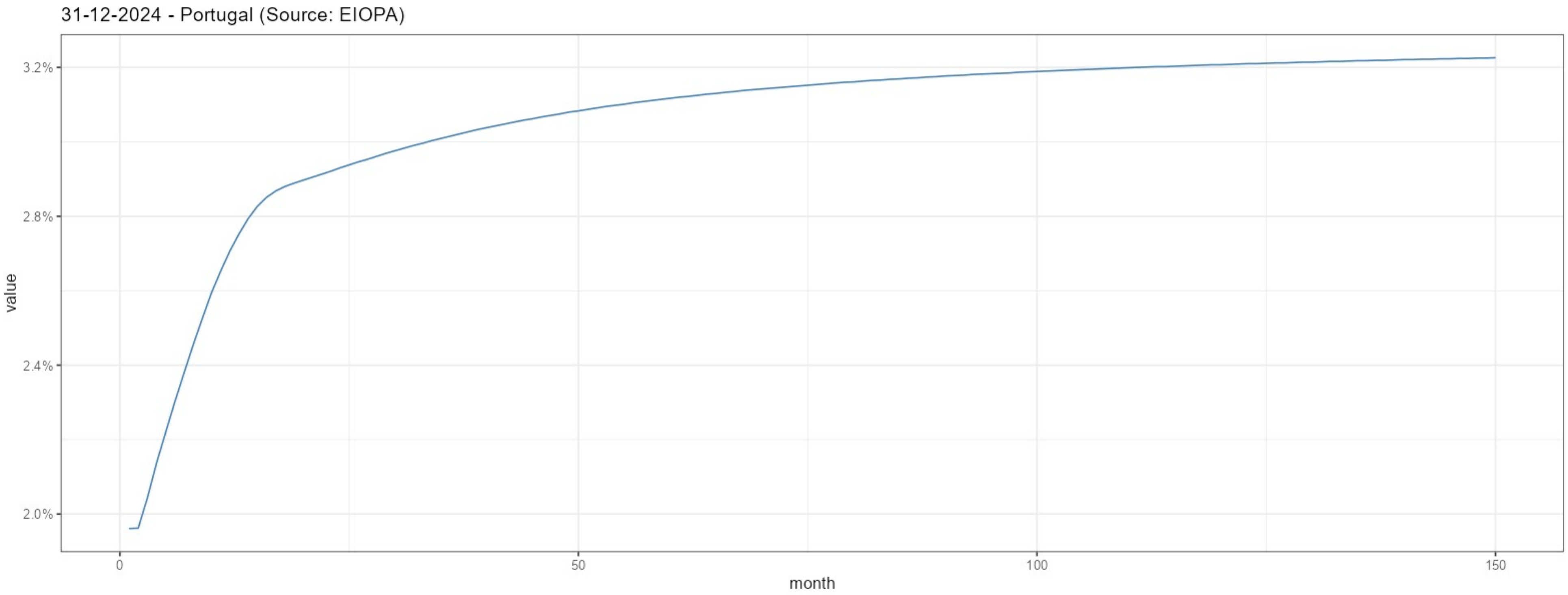
> **Note**
>
> The `yield` is just a way of expressing the `price`.

- $y(t)$ is also called the `spot rate` or zero rate for maturity t.
- the `current` yield curve describes the yields of `notional` zero-coupon bonds of €1, due at different times in the future.
- `Current` because the market price changes every day.
- `Notional` because there aren't zero coupon bonds for every maturity.

Yield curve examples

(see R script 1.Example yields.R)

Portugal



Discounting

- Assume that the yield curve $\{y(t) : t > 0\}$ is known.
- The arbitrage-free market value of a risk-free, future cashflow $\{c(t_1), c(t_2), \ldots, c(t_n)\}$ is:

$$B = \sum_{i=1}^{n} P(t_i)\, c(t_i) = \sum_{i=1}^{n} e^{-y(t_i)t_i} c(t_i)$$

- Every payment is valued separately as a zero-coupon bond.

Yields are strange

Consider this:

- The spot rate $y(t)$ at maturity t is the `constant` yield rate in the interval $(0, t)$ that reproduces the observed price $P(t)$ of €1 payable at time t.
- At the same time we are aware that the yield curve is `not constant`!

Forward rates

- The `forward rate` $y_F(t)$ is the implied yield in the infinitesimal time interval $(t, t + dt)$, defined consistently with the spot rate.
- The spot rate is the `average` of forward rates in the interval $(0, t)$.

Forward rates

- Forward rates $y_F(t)$ are defined by spot rates through the equation

$$\int_0^t y_F(s)ds = y(t) \cdot t.$$

- Assuming differentiability, we have

$$y_F(t) = y(t) + t \cdot y'(t).$$

Annual compounding

- Let n be an integer.
- Let $P(n)$ be the market price of a `zero coupon bond` that pays the risk free amount of €1 at time n.
- Then the yield i with annual compounding is defined by

$$P(n) = (1 + i)^{-n}.$$

- The yield of zero coupon bonds can be explicitly calculated:

$$i = i(n) = P(n)^{-\frac{1}{n}} - 1 = e^{y(n)} - 1$$

Annual compounding

Note

Recall the relationship between yield with annual compounding (i) and yield with continuous compounding (y):

$$i = e^y - 1$$

$$y = ln(1 + i)$$

Why continuous compounding?

- Continuous compounding allows a `unified and simple notation`, e.g.

$$P(t) = exp(-y(t) \cdot t) = exp\left(\int_0^t y_F(s)ds\right)$$

  regardless of wether t is an integer (whole year) or not.

- In this lecture we will use continuous compounding.
- In the financial press, annual and semi-annual compounding is common.

Bonds

- A bond can be defined in general as *"a promise to make a series of payments of specified size, at specified times in the future"*.
- Let us denote by $c(t_i)$ the payment due at time t_i, for $i = 1, \ldots, n$.
- We assume that bonds have no credit risk.

Bond yield

- Let $\{c(t_i) : i = 1, \ldots, n\}$ be the payments stipulated by a bond.
- Let B be the price being paid for the bond in the market.
- The average yield $\bar{y}$ of the bond is defined (implicitly) by

$$B = B(\bar{y}) \stackrel{!}{=} \sum_{i=1}^{n} e^{-\bar{y} t_i c(t_i)} \stackrel{def}{=} \int_0^\infty e^{-\bar{y} t} dC(t)$$

- The average bond yield is well-defined if all payments are non-negative.

Bond yield example

We are at the 31st December 2024. We will compute `forward rates` compatible with the (continuous) assumed market yield, the `price of the bond` and its `yield` (annual and continuous).

- Face value: 100
- Annual coupons: 5%
- Maturity: 5 years
- Market assumptions for Portugal by EIOPA

Coupon (%) 5

Face Value 100

time t	spot rate	price of €1	cash flows	PV c.flows
0.00	0.000%	1.00	0.00	0.00
1.00	2.707%	0.97	5.00	4.87
2.00	2.930%	0.94	5.00	4.72
3.00	3.015%	0.91	5.00	4.57
4.00	3.075%	0.88	5.00	4.42
5.00	3.117%	0.86	105.00	89.85

- Bond price: 108.4179946

Yield curve estimation

Estimating the market yield curve by `replication`

- Assume that you know the market prices $B_1, \ldots, B_n$ of n different government bonds.
- Define the payoff matrix

$$\mathbf{C} = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix} = \begin{pmatrix} Payments\ of\ bond\ 1 \\ \vdots \\ Payments\ of\ bond\ n \end{pmatrix}$$

- Some of the c_{ij} may be `zero` but all bonds' total payments must be restricted to the time points $t_1, \ldots, t_n$.

Yield curve estimation - replication

- We construct a portfolio $(w_1, \ldots, w_n)$ that `replicates` the cash flow of a zero-coupon bond at maturity t_j:

$$(w_1, \ldots, w_n)\mathbf{C} \stackrel{!}{=} (0, \ldots, 0, 1, 0, \ldots, 0)$$

- The equation is solved by

$$(w_1, \ldots, w_n) = (0, \ldots, 0, 1, 0, \ldots, 0)\mathbf{C}^{-1} = row_j\mathbf{C}^{-1}$$

- Then, the price of the zero-coupon bond at maturity t_j is

$$P(t_j) = \sum_{i=1}^{n} w_i B_i$$

Yield curve estimation - replication

- The `implied zero rate` $y(t_j)$ is given by solving

$$P(t_j) = e^{y(t_j)t_j}$$

- In theory, finding yield curves is easy matrix algebra. In practice there are a number of problems. For example:
  - Not enough traded bonds to cover all time points.
  - Payments at other time points.
  - Lack of long term bonds.
- In practice you would use a software or the `risk-free rates delivered by EIOPA, Bloomberg` or others.

Example - Market assumption 31/12/2024

```
# A tibble: 15 × 5
    Bond `Mat. 31/12` `Face value` `Face val.` `Avg. yield`
   <dbl>        <dbl>        <dbl>       <dbl>        <dbl>
 1     1         2025          100        0.04       0.0219
 2     2         2026          100        0.04       0.0247
 3     3         2027          100        0.04       0.0255
 4     4         2028          100        0.05       0.0267
 5     5         2029          100        0.05       0.0281
 6     6         2030          100        0.05       0.0293
 7     7         2031          100        0.05       0.0305
 8     8         2032          100        0.05       0.0315
 9     9         2033          100        0.05       0.0324
10    10         2034          100        0.05       0.0329
11    11         2035          100        0.05       0.0332
```

Example - Payment Matrix

	2024	2025	2026	2027	2028	2029	2030	2031	2032	2033	2034	2035	2036	2037
[1,]	104	0	0	0	0	0	0	0	0	0	0	0	0	0
[2,]	4	104	0	0	0	0	0	0	0	0	0	0	0	0
[3,]	4	4	104	0	0	0	0	0	0	0	0	0	0	0
[4,]	5	5	5	105	0	0	0	0	0	0	0	0	0	0
[5,]	5	5	5	5	105	0	0	0	0	0	0	0	0	0
[6,]	5	5	5	5	5	105	0	0	0	0	0	0	0	0
[7,]	5	5	5	5	5	5	105	0	0	0	0	0	0	0
[8,]	5	5	5	5	5	5	5	105	0	0	0	0	0	0
[9,]	5	5	5	5	5	5	5	5	105	0	0	0	0	0
[10,]	5	5	5	5	5	5	5	5	5	105	0	0	0	0
[11,]	5	5	5	5	5	5	5	5	5	5	105	0	0	0
[12,]	5	5	5	5	5	5	5	5	5	5	5	105	0	0
[13,]	5	5	5	5	5	5	5	5	5	5	5	5	105	0

Example - Clean market price B

```
# A tibble: 15 × 6
    Bond `Maturity 31.12. …` `Face value` Coupon `Average yield_annual`
   <dbl>                <dbl>        <dbl>  <dbl>                  <dbl>
 1     1                 2025          100   0.04                 0.0219
 2     2                 2026          100   0.04                 0.0247
 3     3                 2027          100   0.04                 0.0255
 4     4                 2028          100   0.05                 0.0267
 5     5                 2029          100   0.05                 0.0281
 6     6                 2030          100   0.05                 0.0293
 7     7                 2031          100   0.05                 0.0305
 8     8                 2032          100   0.05                 0.0315
 9     9                 2033          100   0.05                 0.0324
10    10                 2034          100   0.05                 0.0329
11    11                 2035          100   0.05                 0.0332
```

Example - market yield curve

	time	price of €1	spot rate
1	1	0.9785475	2.169%
2	2	0.9522118	2.448%
3	3	0.9269405	2.529%
4	4	0.8994891	2.648%
5	5	0.8693488	2.800%
6	6	0.8390086	2.926%
7	7	0.8074717	3.055%
8	8	0.7765943	3.160%
9	9	0.7453737	3.265%
10	10	0.7180624	3.312%
11	11	0.6922223	3.344%
12	12	0.6669583	3.375%
13	13	0.6423107	3.405%

Yield curve estimation - bootstrapping

- Assume you have bonds $i = 1, \ldots, n$.
- Bond nr. i matures at time t_i, pays coupon c_i and its current market price is B_i.
- All bonds have principal 1.

1. Solve for the first bond

$$B_1 = (1 + c_1)\, P(t_1) \Rightarrow P(t_1) = \frac{B_1}{1 + c_1} = e^{-y(t_1)t_1}$$

Yield curve estimation - bootstrapping

2. Solve for each subsequent bond

$$B_m = c_m \underbrace{\sum_{i=1}^{m-1} P(t_i)}_{\textbf{known}} + (1 + c_m)\ \underbrace{P(t_m)}_{\textbf{unknown}}$$

$$\Rightarrow \quad P(t_m) = \frac{B_m - c_m \sum_{i=1}^{m-1} P(t_i)}{1 + c_m} = e^{-y(t_m)t_m}$$

Present value sensitivity

- Let's assume we have
  - a future cash flow $\{C(t) : t > 0\}$;
  - the current yield curve $\{y(t) : t > 0\}$.
- The present value of the cash flow is

$$B(y) = \int_0^\infty e^{-y(t)t} dC(t)$$

- How will the **PV** of *B* change if the yield curve changes?
- The easy answer: Calculate it!
- The traditional answer: Estimate it!

Duration and convexity 1

- The derivative of the PV with respect to a uniform shift in the entire yield curve is

$$B'(y) = \lim_{\Delta\bar{y}\to 0} \frac{1}{\Delta\bar{y}} \left(\int_0^\infty e^{-(y(t)+\Delta\bar{y})t} dC(t) - \int_0^\infty e^{-y(t)t} dC(t) \right)$$

- The first and second derivative of the PV are

$$B'(y) = -\int_0^\infty t e^{-y(t)t} dC(t),\ B''(y) = -\int_0^\infty t^2 e^{-y(t)t} dC(t)$$

Duration and convexity 2

- Using the `Taylor expansion` we approximate the change in present value if the yield curve shifts:

$$B(y+\Delta\bar{y}) - B(y) \approx B'(y)\Delta\bar{y} + \frac{1}{2}B''(y)(\Delta\bar{y})^2$$

- Define `duration` of the cash flow as

$$D = D(y) = -B'(y)/B(y)$$

- Define `convexity` of the cash flow as

$$C = C(y) = B''(y)/B(y)$$

Duration and convexity 3

- Rewrite the Taylor expansion in the following way:

$$\frac{B(y+\Delta\bar{y})-B(y)}{B(y)} \approx -D(y)\Delta\bar{y}+\frac{1}{2}C(y)(\Delta\bar{y})^2$$

- In words: One can approximate the `relative change` in the **PV** of the cash flow when the yield curve is shifted `uniformly` by a small amount.
  - To first order: minus the yield change $\Delta\bar{y}$, times duration.
  - To second order: Same as above, plus the squared yield change times one-half convexity.

Example PV Sensitivity 1

Consider a bond with face value of €100, maturity of 5 years and yearly coupons of 5%.

```
  duration convexity
1 4.567348  21.98331
```

- We will value it under market assumptions (€110.07) and estimate the effect of a parallel yield perturbation:
    - increase of 1% - `1st. order` €105.04, `2nd order` €105.16.
    - decrease of 1% - `1st. order` €115.09, `2nd order` €115.22

Properties of duration and convexity 1

- The duration and convexity of a `zero-coupon bond payable at time t` are t and t^2, independent of the yield.
- Duration and convexity decrease when the yield increases.
- For a given duration, convexity increases with the dispersion of the flow, because

$$\underbrace{\frac{1}{B(y)}\int_0^\infty (t - D(y))^2 e^{-y(t)t} dC(t)}_{\textbf{Dispersion, similar to variance}} = C(y) - D^2(y)$$

Properties of duration and convexity 2

- The duration/convexity approximation is an `easy way to estimate the sensitivity` of a cash flow's **PV** to small changes in the yield curve.
- The average duration/convexity of a portfolio is the **PV**-weighted average of the constituent durations/convexities. This makes those quantities easy to use.
- The duration/convexity approximation is `valid only` when there is a `parallel` shift in the yield curve.

Properties of duration and convexity 3

- The duration/convexity approximation `does not` tell us what change in the present value to expect, should different parts of the yield curve change by different amounts or even in different directions.

Different concepts of duration

- `Macaulay Duration`: The time weighted PV divided by the PV.
- `Modified Duration`: Macaulay Duration divided by $1 + i(n)/n$, where n is the compounding frequency.
- `Effective Duration`: Calculated by shocking the yield curve up and down by some change in PV.
- `Dollar Duration`: $DD(y) = -B'(y) = B(y)D(y)$
- `Dollar Convexity`: $DC(y) = B''(y) = B(y)C(y)$